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On minimax modules and generalized local cohomology with respect to a pair of ideals

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Abstract We study the minimax properties and the artinianness of the generalized local cohomology modules $H^i_{I,J}(M,N)$ with respect to a pair of ideals (I,J). We also show some results on top generalized local cohomology modules.

Keywords Attached primes · Generalized local cohomology · Minimax

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1 Introduction

Throughout this paper, R is a noetherian commutative (with non-zero identity) ring. It is well-known that the local cohomology theory of Grothendieck is an important tool in commutative algebra and algebraic geometry. There are some generalizations of this theory, one of them was introduced by Herzog [10]. He defined the generalized local cohomology modules $H^i_I(M,N)$ of M

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and N with respect to I by

$$H^i_I(M,N) = \varinjlim_n \operatorname{Ext}^i_R(M/I^nM, N).$$

In 2009, Takahashi, Yoshino and Yoshizawa introduced local cohomology modules with respect to a pair of ideals (I,J) [17]. For an R-module M, the (I,J)-torsion submodule of M is $\varGamma_{I,J}(M) = \{x \in M \mid I^n x \subseteq Jx \text{ for some positive integer } n\}$. They denoted by $H^i_{I,J}$ the i-th right derived functor of the functor $\varGamma_{I,J}$. It is clear that when $J = 0$, the functor $H^i_{I,0}$ coincides with the usual local cohomology functor H^i_I.

In [14], for two R-modules M and N, we defined $\varGamma_{I,J}(M,N)$ to be the (I,J)-torsion submodule of $\operatorname{Hom}_R(M,N)$. For each R-module M, there is a covariant functor $\varGamma_{I,J}(M,-)$ from the category of R-modules to itself. The i-th generalized local cohomology functor $H^i_{I,J}(M,-)$ with respect to a pair of ideals (I,J) is the i-th right derived functor of the functor $\varGamma_{I,J}(M,-)$. This definition is really a generalization of the local cohomology functor $H^i_{I,J}$. On the other hand, $H^i_{I,J}(M,N)$ is also an extension of the generalized local cohomology module $H^i_I(M,N)$. Also in [14], we study some basic properties of $H^i_{I,J}(M,N)$.

The organization of the paper is as follows. In Section 2, we study the minimax properties of $H^i_{I,J}(M,N)$. It should be mentioned that the minimax modules were introduced by H. Zöschinger [19]. Let M be a finitely generated R-module. The first result is Theorem 1 which says that if N and $H^i_{I,J}(N)$ are minimax for all $i < t$, then so is $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M,N))$ for all $\mathfrak{a} \in \tilde{W}(I,J)$.

In Section 3 we proceed with the study of the artinianness of $H^i_{I,J}(M,N)$. In Proposition 1, if N is a minimax module, then $H^{p+d}_{I,J}(M,N)$ is an artinian module, where $p = \operatorname{pd}M$ and $d = \dim N$. Theorem 2 gives us the isomorphism $H^i_{I,J}(M,N) \cong H^i_{\mathfrak{m},J}(M,N)$ provided $\operatorname{Supp}_R(H^i_{I,J}(M,N)) \subseteq \{\mathfrak{m}\}$. In the case $(R,\mathfrak{m})$ is a local ring and N is a minimax module such that $\operatorname{Supp}_R(H^i_{I,J}(M,N)) \subseteq \{\mathfrak{m}\}$ we prove that $H^i_{I,J}(M,N)$ is artinian. One of our main results is Theorem 3 which says that in a local ring $(R,\mathfrak{m})$, the module $H^i_{I,J}(M,N)$ is artinian for all $i \le t$ if and only if $H^i_{\mathfrak{a}}(M,N)$ is artinian for all $i \le t$ and $\mathfrak{a} \in \tilde{W}(I,J)$.

The last section is devoted to the study of top generalized local cohomology modules. One of the basic problems concerning local cohomology is to find the set of attached prime ideals of local cohomology modules of finitely generated R-modules. Let $(R,\mathfrak{m})$ be a local ring and M,N finitely generated R-modules with $\operatorname{pd}M = p < \infty, \dim N = d < \infty$. Theorem 4 gives us the set of attached prime ideals of top generalized local cohomology module $H^{p+d}_{I,J}(M,N)$ as follows

$$\operatorname{Att}(H^{p+d}_{I,J}(M,N)) = \{\mathfrak{p} \in \operatorname{Ass}_R(N/JN) \mid \operatorname{cd}(I,M,R/\mathfrak{p}) = p+d\}.$$

Finally, Proposition 3 shows that if $I+J$ is $\mathfrak{m}$-primary and $\dim N = d > 0$, then $H^{p+d}_{I,J}(M,N) = 0$ or $H^{p+d}_{I,J}(M,N)$ is not finitely generated.

2 The minimax properties of $H^i_{I,J}(M,N)$

Let I, J be two ideals of R. In [17] they denoted by $W(I,J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I^n \subseteq \mathfrak{p}+J, \text{ for some } n \gg 1\}$ and $\tilde{W}(I,J) = \{\mathfrak{a} \lhd R \mid I^n \subseteq \mathfrak{a}+J, \text{ for some } n \gg 1\}$.

For two R-modules M and N, in [14] we defined

$$\Gamma_{I,J}(M,N) = \Gamma_{I,J}(\operatorname{Hom}_R(M,N)).$$

In the special case $M = R, \Gamma_{I,J}(R,N) = \Gamma_{I,J}(N)$ the (I,J)-torsion submodule of N. For each R-module M, $\Gamma_{I,J}(M,-)$ is a left exact covariant functor from the category of R-modules to itself and the i-th right derived functor of $\Gamma_{I,J}(M,-)$ is called the i-th generalized local cohomology functor $H^i_{I,J}(M,-)$ with respect to a pair of ideals (I,J).

Lemma 1 ([14, 2.2]) *Let M be a finitely generated R-module and N an R-module. Then*

$$\Gamma_{I,J}(M,N) = \operatorname{Hom}_R(M, \Gamma_{I,J}(N)).$$

Lemma 2 ([14, 2.6]) *Let N be an (I,J)-torsion R-module. Then*

$$H^i_{I,J}(M,N) \cong \operatorname{Ext}^i_R(M,N)$$

for all $i \geq 0$.

We recall some facts about minimax modules. An R-module M is said to be minimax if M has a finitely generated submodule L such that M/L is artinian ([19]). The class of minimax R-modules is a Serre subcategory. In particular, let M be a finitely generated R-module and N a minimax R-module. Then for any integer i, the modules $\operatorname{Tor}^R_i(M,N)$ and $\operatorname{Ext}^i_R(M,N)$ are minimax.

Lemma 3 ([11, 2.1]) *Let I be an ideal of the ring R and M a minimax R-module. Then M is I-torsion-free if and only if I contains a non-zerodivisor on M.*

The following theorem concerns with the minimaxness of $H^i_{I,J}(N)$ and $H^i_{I,J}(M,N)$, note that the module N is not necessarily finitely generated.

Theorem 1 *Let M be a finitely generated R-module and N an R-module. If t is a non-negative integer such that $H^i_{I,J}(N)$ is minimax for all $i < t$, then*

(i) $\operatorname{Ext}^i_R(R/\mathfrak{a},N)$ *is minimax for all $i < t$ for all $\mathfrak{a} \in \tilde{W}(I,J)$.*
(ii) *We assume in addition that* $\operatorname{Ext}^t_R(R/\mathfrak{a},N)$ *is minimax for all $\mathfrak{a} \in \tilde{W}(I,J)$, then* $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(N))$ *is minimax.*
(iii) *In the case N is minimax, then* $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M,N))$ *is also minimax for all $\mathfrak{a} \in \tilde{W}(I,J)$.*

Proof (i) We use induction on t. When $t = 0$, it is nothing to prove.

Let $t > 0$ and $\overline{N} = N/\Gamma_{I,J}(N)$, the exact sequence

$$0 \to \Gamma_{I,J}(N) \to N \to \overline{N} \to 0$$

induces a long exact sequence

$$\cdots \operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_{I,J}(N)) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, N) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \overline{N}) \cdots .$$

Since $\Gamma_{I,J}(N)$ is minimax, the proof of (i) is completed by showing that $\operatorname{Ext}^i_R(R/\mathfrak{a}, \overline{N})$ is minimax for all $i < t$. By virtue of [17, 1.13], $H^i_{I,J}(N) \cong H^i_{I,J}(\overline{N})$ for all $i > 0$ and $\Gamma_{I,J}(\overline{N}) = 0$.

Denote by $E = E(\overline{N})$ the injective hull of $\overline{N}$ and $L = E/\overline{N}$. Note that $\operatorname{Ass}_R(\Gamma_{I,J}(E)) = W(I,J) \cap \operatorname{Ass}_R(E) = W(I,J) \cap \operatorname{Ass}_R(\overline{N}) = \emptyset$, so $\Gamma_{I,J}(E) = 0$. The short exact sequence

$$0 \to \overline{N} \to E \to L \to 0$$

induces the following isomorphisms

$$H^i_{I,J}(L) \cong H^{i+1}_{I,J}(\overline{N}) \text{ and } \operatorname{Ext}^i_R(R/\mathfrak{a}, L) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, \overline{N})$$

for all $i \geq 0$.

It follows from the inductive hypothesis that $\operatorname{Ext}^i_R(R/\mathfrak{a}, L)$ is minimax for all $i < t-1$. Therefore $\operatorname{Ext}^i_R(R/\mathfrak{a}, \overline{N})$ is minimax for all $i < t$ and the proof of (i) is complete.

(ii) The proof is by induction on t. When $t = 0$, note that $\overline{N}$ is $\mathfrak{a}$-torsion-free, so $\operatorname{Hom}_R(R/\mathfrak{a}, \overline{N}) = 0$. Applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ to the exact sequence

$$0 \to \Gamma_{I,J}(N) \to N \to \overline{N} \to 0$$

we conclude that $\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{I,J}(N)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, N)$ and then $\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{I,J}(N))$ is minimax.

Assume that $t > 0$. It follows from the proof of (i), we have

$$\begin{aligned} \operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{I,J}(L)) &\cong \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(\overline{N})) \\ &\cong \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(N)). \end{aligned}$$

By the inductive hypothesis, $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{I,J}(L))$ is minimax and then $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(N))$ is mininax.

(iii) We use induction on t. When $t = 0$, we have $H^0_{I,J}(M,N) \cong \operatorname{Hom}_R(M, \Gamma_{I,J}(N))$ by 1. Since $\Gamma_{I,J}(N) \subseteq N$, $\operatorname{Hom}_R(M, \Gamma_{I,J}(N))$ is minimax and then $\operatorname{Hom}_R(R/\mathfrak{a}, H^0_{I,J}(M,N))$ is also minimax.

Let $t > 0$. The short exact sequence

$$0 \to \Gamma_{I,J}(N) \to N \to \overline{N} \to 0$$

induces a long exact sequence

$$\cdots \to H^i_{I,J}(M, \Gamma_{I,J}(N)) \to H^i_{I,J}(M,N) \to H^i_{I,J}(M,\overline{N}) \to \cdots .(*)$$

It follows from Lemma 2 that $H^i_{I,J}(M, \Gamma_{I,J}(N)) \cong \operatorname{Ext}^i_R(M, \Gamma_{I,J}(N))$ for all i, then $H^i_{I,J}(M, \Gamma_{I,J}(N))$ is minimax for all i. By virtue of the exact sequence (*), the proof will be finished if we prove that $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M, \overline{N}))$ is minimax. Since $\overline{N}$ is (I,J)-torsion-free, it is also $\mathfrak{a}$-torsion-free for all $\mathfrak{a} \in \tilde{W}(I,J)$. By Lemma 3 there exists an $\overline{N}$-regular element $x \in \mathfrak{a}$. Now the short exact sequence

$$0 \to \overline{N} \xrightarrow{\cdot x} \overline{N} \to \overline{N}/x\overline{N} \to 0$$

gives rise a long exact sequence

$$\cdots H^{t-1}_{I,J}(M, \overline{N}) \xrightarrow{f} H^{t-1}_{I,J}(M, \overline{N}/x\overline{N}) \xrightarrow{g} H^t_{I,J}(M, \overline{N}) \xrightarrow{\cdot x} H^t_{I,J}(M, \overline{N}) \cdots .$$

Since $H^i_{I,J}(\overline{N}) \cong H^i_{I,J}(N)$ for all $i > 0$, it follows from [15, 2.5] that $H^i_{I,J}(M, \overline{N})$ is minimax for all $i < t$. Thus $H^i_{I,J}(M, \overline{N}/x\overline{N})$ is minimax, so $H^i_{I,J}(\overline{N}/x\overline{N})$ is minimax for all $i < t-1$. Note that $\overline{N}/x\overline{N}$ is also minimax. The inductive hypothesis shows that $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{I,J}(M, \overline{N}/x\overline{N}))$ is minimax.

Splitting the above exact sequence, we obtain two exact sequences

$$0 \to \operatorname{Im} f \to H^{t-1}_{I,J}(M, \overline{N}/x\overline{N}) \to \operatorname{Im} g \to 0,$$

$$0 \to \operatorname{Im} g \to H^t_{I,J}(M, \overline{N}) \xrightarrow{\cdot x} H^t_{I,J}(M, \overline{N}).$$

Applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ to these exact sequences, we get the following exact sequences

$$\begin{aligned} 0 \to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} f) &\to \operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{I,J}(M, \overline{N}/x\overline{N})) \\ &\to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} g) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, \operatorname{Im} f) \cdots , \end{aligned}$$

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} g) \to \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M, \overline{N})) \xrightarrow{\cdot x} \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M, \overline{N})).$$

Since $H^{t-1}_{I,J}(M, \overline{N})$ is a minimax module, $\operatorname{Im} f$ is also a minimax module. Then $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} g)$ is also minimax. As $x \in \mathfrak{a}$, the last exact sequence gives an isomorphism

$$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} g) \cong \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M, \overline{N})).$$

Therefore $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M, \overline{N}))$ is minimax and the proof of (iii) is complete.

Corollary 1 *Let M be a finitely generated R-module, N an R-module. Let t be a non-negative integer such that $H^i_{I,J}(N)$ is minimax for all $i < t$.*

(i) *Assume that for each $\mathfrak{a} \in \tilde{W}(I,J)$, $\operatorname{Ext}^t_R(R/\mathfrak{a}, N)$ is minimax and there is a finitely generated R-module K such that $\operatorname{Supp}_R(K) \subseteq V(\mathfrak{a})$. Then $\operatorname{Hom}_R(K, H^t_{I,J}(N))$ is a minimax module.*

(ii) *If N is minimax and K is a submodule of $H^t_{I,J}(M, N)$ such that $\operatorname{Ext}^1_R(R/\mathfrak{a}, K)$ is minimax, then $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M, N)/K)$ is also minimax.*

Proof (i) As $\mathrm{Supp}_R(K) \subseteq V(\mathfrak{a})$, there exists a finite chain $0 = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = K$ of submodules of K such that K_i/K_{i-1} is a homomorphic image of finitely many copies of $R/\mathfrak{a}$ for all $i = 1, 2, \ldots, n$ by Gruson's theorem. Then we have a short exact sequence

$$0 \to L_1 \to (R/\mathfrak{a})^{m_1} \to K_1 \to 0,$$

where L_1 is an R-module and m_1 is a positive integer. It deduces a long exact sequence

$$0 \to \mathrm{Hom}_R(K_1, H^t_{I,J}(N)) \to \mathrm{Hom}_R((R/\mathfrak{a})^{m_1}, H^t_{I,J}(N)) \to \mathrm{Hom}_R(L_1, H^t_{I,J}(N)).$$

We also have a short exact sequence

$$0 \to L_2 \to (R/\mathfrak{a})^{m_2} \to K_2/K_1 \to 0.$$

Applying the functor $\mathrm{Hom}_R(-, H^t_{I,J}(N))$ to this short exact sequence, we get an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(K_2/K_1, H^t_{I,J}(N)) &\to \mathrm{Hom}_R((R/\mathfrak{a})^{m_2}, H^t_{I,J}(N)) \\ &\to \mathrm{Hom}_R(L_2, H^t_{I,J}(N)) \cdots \end{aligned}$$

Then $\mathrm{Hom}_R(K_1, H^t_{I,J}(N))$ and $\mathrm{Hom}_R(K_2/K_1, H^t_{I,J}(N))$ are minimax by Theorem 1 (ii). By using induction on n, it follows that $\mathrm{Hom}_R(K_{n-1}, H^t_{I,J}(N))$ and $\mathrm{Hom}_R(K/K_{n-1}, H^t_{I,J}(N))$ are minimax modules. Therefore $\mathrm{Hom}_R(K, H^t_{I,J}(N))$ is minimax for all finitely generated R-module with $\mathrm{Supp}_R(K) \subseteq V(\mathfrak{a})$.

(ii) Applying the functor $\mathrm{Hom}_R(R/\mathfrak{a}, -)$ to the short exact sequence

$$0 \to K \to H^t_{I,J}(M, N) \to H^t_{I,J}(M, N)/K \to 0$$

we obtain a long exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M, N)) \to \mathrm{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M, N)/K) \to \mathrm{Ext}^1_R(R/\mathfrak{a}, K)$$

It follows from Theorem 1 (iii) that $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M, N))$ is minimax. Hence the conclusion follows from the hypothesis.

3 The artinianness of $H^i_{I,J}(M, N)$

In [14, 3.1], we proved that if $(R, \mathfrak{m})$ is a local ring and M, N are finitely generated R-modules with $\dim N = d$ and $\mathrm{pd} M = p$, then $H^{p+d}_{I,J}(M, N)$ is an artinian R-module. We have a generalization of [14, 3.1] for minimax modules in the following theorem.

Proposition 1 *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module with $\mathrm{pd} M = p$. If N is a minimax R-module with $\dim N = d$, then $H^{p+d}_{I,J}(M, N)$ is an artinian R-module.*

Proof Since N is a minimax R-module, we have a short exact sequence

$$0 \to L \to N \to A \to 0,$$

where L is finitely generated and A is artinian. It induces a long exact sequence

$$\ldots \to H^i_{I,J}(M, L) \to H^i_{I,J}(M, N) \to H^i_{I,J}(M, A) \to \ldots.$$

From [14, 3.7], $H^i_{I,J}(M, A)$ is an artinian R-module for all $i \geq 0$. Combining [14, 3.2] with [18, 2.4] we conclude that $H^{p+d}_{I,J}(M, L)$ is an artinian R-module since $\dim L \leq \dim N = d$. Therefore $H^{p+d}_{I,J}(M, N)$ is also an artinian R-module.

In [6], when M, N are finitely generated R-modules and $(R, \mathfrak{m})$ is a local ring, Chu and Tang showed that $\mathrm{Supp}_R(H^i_I(M, N)) \subseteq \{\mathfrak{m}\}$ if and only if $H^i_I(M, N)$ is artinian. In the following theorem, we generalize this result in the case of $H^i_{I,J}(M, N)$.

Theorem 2 *Let $(R, \mathfrak{m})$ be a local ring and t a non-negative integer. Assume that M is a finitely generated R-module and N is an R-module such that $\mathrm{Supp}_R(H^i_{I,J}(M, N)) \subseteq \{\mathfrak{m}\}$ for all $i \leq t$.*

(i) *Then $H^i_{I,J}(M, N) \cong H^i_{\mathfrak{m},J}(M, N) \cong H^i_{\mathfrak{m}}(M, N)$ for all $i \leq t$.*
(ii) *If N is minimax, then $H^i_{I,J}(M, N)$ is artinian for all $i \leq t$.*

Proof (i) First we prove $H^i_{I,J}(M, N) \cong H^i_{\mathfrak{m},J}(M, N)$ for all $i \leq t$. Denote by $F(-) := \Gamma_{\mathfrak{m},J}(-)$ and $G(-) := \Gamma_{I,J}(M, -)$ functors from the category of R-modules to itself. Note that F is a left exact functor. For each R-module N, we have

$$\begin{aligned} FG(N) &= \Gamma_{\mathfrak{m},J}(\Gamma_{I,J}(M, N)) \\ &= \Gamma_{\mathfrak{m},J}(\Gamma_{I,J}(\mathrm{Hom}_R(M, N))) \\ &= \Gamma_{\mathfrak{m}+I,J}(\mathrm{Hom}_R(M, N)) \\ &= \Gamma_{\mathfrak{m},J}(\mathrm{Hom}_R(M, N)) \\ &= \Gamma_{\mathfrak{m},J}(M, N). \end{aligned}$$

Hence $FG(-) = \Gamma_{\mathfrak{m},J}(M, -)$.

Let E be an injective R-module. Since $\Gamma_{I,J}(E)$ is an injective module, $R^iF(G(E)) = R^iF(\Gamma_{I,J}(M, E)) \cong R^iF(\mathrm{Hom}_R(M, \Gamma_{I,J}(E))) = 0$ for all $i > 0$. By [16, 10.47] there is a Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{m},J}(H^q_{I,J}(M, N)) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{m},J}(M, N).$$

As $\mathrm{Supp}_R(H^q_{I,J}(M, N)) \subseteq \{\mathfrak{m}\} \subseteq W(\mathfrak{m}, J)$, $H^q_{I,J}(M, N)$ is an $(\mathfrak{m}, J)$-torsion module for all $0 \leq q \leq t$. By [17, 1.13]

$$E_2^{p,q} = H^p_{\mathfrak{m},J}(H^q_{I,J}(M, N)) = 0$$

for all $p > 0, 0 \leq q \leq t$.

Let $0 \leq n \leq t$, there is a filtration Φ of $H^n = H^n_{\mathfrak{m},J}(M,N)$ such that

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \ldots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n_{\mathfrak{m},J}(M,N)$$

and

$$E^{i,n-i}_\infty \cong \Phi^i H^n/\Phi^{i+1}H^n, 0 \leq i \leq n.$$

In particular

$$E^{0,n}_\infty \cong \Phi^0 H^n/\Phi^1 H^n,$$

$$E^{i,n-i}_\infty \cong \Phi^i H^n/\Phi^{i+1}H^n = 0, 0 < i \leq n.$$

Hence $\Phi^1 H^n = \Phi^2 H^n = \ldots = \Phi^{n+1}H^n = 0$ and $E^{0,n}_\infty \cong \Phi^0 H^n = H^n_{\mathfrak{m},J}(M,N)$.

Let $r \geq 2$, from the homomorphisms of the spectral sequence

$$0 = E^{-r,n+r-1}_r \to E^{0,n}_r \to E^{r,n-r+1}_r = 0$$

we get that $E^{0,n}_2 = E^{0,n}_3 = \ldots = E^{0,n}_\infty$. Thus

$$H^n_{\mathfrak{m},J}(M,N) \cong E^{0,n}_2 = H^0_{\mathfrak{m},J}(H^n_{I,J}(M,N)) = H^n_{I,J}(M,N)$$

as $H^n_{I,J}(M,N)$ is an $(\mathfrak{m},J)$-torsion module.

Next, to prove $H^i_{I,J}(M,N) \cong H^i_{\mathfrak{m}}(M,N)$ for all $i \leq t$, we apply the above argument with $F(-) := \Gamma_{\mathfrak{m},J}(-)$ replaced by $F(-) := \Gamma_{\mathfrak{m}}(-)$.

(ii) Combining [8, 2.2] and (i), we see that $H^i_{I,J}(M,N)$ is artinian for all $i \leq t$ when N is finitely generated.

We now assume that N is a minimax module. Then there is a short exact sequence

$$0 \to L \to N \to A \to 0,$$

where L is finitely generated and A is artinian. It induces a long exact sequence

$$\ldots \to H^i_{I,J}(M,L) \to H^i_{I,J}(M,N) \to H^i_{I,J}(M,A) \to \ldots.$$

Note that $\mathrm{Supp}_R(H^i_{I,J}(M,A)) \subseteq \{\mathfrak{m}\}$ because $H^i_{I,J}(M,A)$ is artinian by [14, 3.7]. Combining the hypothesis with the long exact sequence, we have $\mathrm{Supp}_R(H^i_{I,J}(M,L)) \subseteq \{\mathfrak{m}\}$ for all $i \leq t$. As L is a finitely generated R-module, $H^i_{I,J}(M,L)$ is artinian for all $i \leq t$ by the above proof. From the long exact sequence $H^i_{I,J}(M,N)$ is artinian for all $i \leq t$. The proof is complete.

We will see a relationship on artinianness of $H^i_{I,J}(M,N)$ and $H^i_I(M,N)$ in the following theorem.

Theorem 3 *Let M be a finitely generated R-module and t a non-negative integer.*

(i) *Then* $\mathrm{Supp}_R(H^i_{I,J}(M,N)) \subseteq \mathrm{Max}(R)$ *for all* $i \leq t$ *if and only if* $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(M,N)) \subseteq \mathrm{Max}(R)$ *for all* $i \leq t$ *and all* $\mathfrak{a} \in \tilde{W}(I,J)$.

(ii) *Assume that* $(R,\mathfrak{m})$ *is a local ring and* N *is finitely generated. Then* $H^i_{I,J}(M,N)$ *is artinian for all* $i \leq t$ *if and only if* $H^i_{\mathfrak{a}}(M,N)$ *is artinian for all* $i \leq t$ *and* $\mathfrak{a} \in \tilde{W}(I,J)$.

Proof (i) ($\Rightarrow$). Let $\mathfrak{a} \in \tilde{W}(I,J)$ and $n \leq t$. Let $F(-) := \Gamma_{\mathfrak{a}}(M,-)$ and $G(-) := \Gamma_{I,J}(-)$ be two functors from the category of R-modules to itself. For an R-module N, we have $FG(M,N) = \Gamma_{\mathfrak{a}}(M,\Gamma_{I,J}(N)) = \Gamma_{\mathfrak{a}}(M,N)$. If E is an injective R-module, then so is $\Gamma_{I,J}(E)$ hence $G(E)$ is right F-acyclic. By [16, 10.47] there is a Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{a}}(H^q_{I,J}(M,N)) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{a}}(M,N)$$

there is a filtration Φ of $H^n = H^n_{\mathfrak{a}}(M,N)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $i \leq n$. Since $E_\infty^{i,n-i}$ is a subquotient of $E_2^{i,n-i}$, we have by the hypothesis that $\mathrm{Supp}_R(E_\infty^{i,n-i}) \subseteq \mathrm{Max}(R)$ for all $i \leq n$. Therefore $\mathrm{Supp}_R(\Phi^i H^n) \subseteq \mathrm{Max}(R)$ for all $i \leq n$. In particular, $\mathrm{Supp}_R(H^n_{\mathfrak{a}}(M,N)) = \mathrm{Supp}_R(\Phi^0 H^n) \subseteq \mathrm{Max}(R)$.

($\Leftarrow$). By [18, 2.2] we see that

$$\mathrm{Supp}_R(H^i_{I,J}(M,N)) \subseteq \bigcup_{\mathfrak{a} \in \tilde{W}(I,J)} \mathrm{Supp}_R(H^i_{\mathfrak{a}}(M,N)).$$

It follows from the hypothesis $\mathrm{Supp}_R(H^i_{I,J}(M,N)) \subseteq \mathrm{Max}(R)$ for all $i \leq t$.

(ii) If $H^i_{I,J}(M,N)$ is artinian, then $\mathrm{Supp}_R(H^i_{I,J}(M,N)) \subseteq \{\mathfrak{m}\}$. It follows from (i) we see that $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(M,N)) \subseteq \{\mathfrak{m}\}$ and then $H^i_{\mathfrak{a}}(M,N))$ is artinian by [6, 2.4] for all $i \leq t$ and all $\mathfrak{a} \in \tilde{W}(I,J)$. The sufficient part is implied by (i) and Theorem 2 (ii).

We now show a result concerning on artinianness of $H^i_{I,J}(N)$ and $H^i_I(M,N)$.

Proposition 2 *Let M be a finitely generated R-module, N an R-module and t a non-negative integer. Assume that $H^i_{I,J}(N)$ is artinian for all $i \leq t$. Then $H^i_I(M,N)$ is also artinian for all $i \leq t$.*

Proof Let $F(-) := \Gamma_I(M,-)$ and $G(-) := \Gamma_{I,J}(-)$ be two left exact functors from the category of R-modules to itself. There is a Grothendieck spectral sequence

$$E_2^{p,q} = H^p_I(M, H^q_{I,J}(N)) \underset{p}{\Rightarrow} H^{p+q}_I(M,N).$$

Let $j \leq t$, there is a filtration Φ of $H^j = H^j_I(M,N)$ such that

$$0 = \Phi^{j+1}H^j \subseteq \Phi^j H^j \subseteq \ldots \subseteq \Phi^1 H^j \subseteq \Phi^0 H^j = H^j_I(M,N)$$

and

$$E_\infty^{i,j-i} \cong \Phi^i H^j / \Phi^{i+1} H^j, 0 \leq i \leq j.$$

Combining the hypothesis with [13, 2.6], $E_2^{p,q}$ is artinian for all $q \leq t$ and for all $p \geq 0$. Since $E_\infty^{i,j-i}$ is a subquotient of $E_2^{i,j-i}$, it follows that $E_\infty^{i,j-i}$ is artinian for all $j - i \leq t$. Consequently, $\Phi^i H^j$ is artinian for all $0 \leq i \leq j$, and the conclusion follows.

4 Top generalized local cohomology modules

We first recall the concept of secondary representation that was introduced by I. G. Macdonald in [12]. An R-module $M \neq 0$ is said to be secondary if, for any $x \in R$, the multiplication by x on M is either surjective or nilpotent. Then $\sqrt{\mathrm{Ann}_R(M)}$ is a prime ideal $\mathfrak{p}$ and we say that M is $\mathfrak{p}$-secondary. A secondary representation of M is an expression of M as finite sum of secondary submodules. If such a representation exists, we will say that M is representable. For the convenience, a zero module is considered as a representable module. A secondary representation of M

$$M = M_1 + \ldots + M_k,$$

where M_i is $\mathfrak{p}_i$-secondary, is said to be minimal if the prime ideals $\mathfrak{p}_i$ are distinct and none of the summands M_i is redundant. Any secondary representation of M can be refined to a minimal one. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ is called the set of attached prime ideals of M and denoted by $\mathrm{Att}_R(M)$. Note that, artinian modules are representable modules ([12]).

In [20], Zöschinger gave another definition of attached prime ideals. Let M be an R-module (not necessarily admitting a secondary representation), a prime ideal $\mathfrak{p}$ of R is said to be an attached prime ideal of M if $\mathfrak{p} = \mathrm{Ann}_R(M/T)$ for some submodule T of M. This definition agrees with the preceding one of attached prime if M admits a secondary representation.

Lemma 4 (See [2]) *The following statements hold true.*

(i) *If $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then*

$$\mathrm{Att}_R C \subseteq \mathrm{Att}_R B \subseteq \mathrm{Att}_R C \cup \mathrm{Att}_R A.$$

(ii) *If N is a finitely generated R-module, then*

$$\mathrm{Att}_R(M \otimes_R N) = \mathrm{Att}_R M \cap \mathrm{Supp}_R N$$

for all R-module M.

In the case $(R, \mathfrak{m})$ is a local ring, it should be noted by [14, 3.1] that $H^{p+d}_{I,J}(M, N)$ is artinian, where $p = \mathrm{pd} M < \infty$ and $d = \dim N < \infty$. Hence, the module $H^{p+d}_{I,J}(M, N)$ has a secondary representation. We now show the set $\mathrm{Att}_R(H^{p+d}_{I,J}(M, N))$ in the following theorem.

Theorem 4 *Let $(R, \mathfrak{m})$ be a local ring and M, N finitely generated R-modules with $p = \mathrm{pd} M < \infty$ and $d = \dim N < \infty$. Then $\mathrm{Att}_R(H^{p+d}_{I,J}(M, N)) = \mathrm{Att}_R(H^{p+d}_{I}(M, N/JN))$ and*

$$\mathrm{Att}_R(H^{p+d}_{I,J}(M, N)) = \{\mathfrak{p} \in \mathrm{Ass}_R(N/JN) \mid \mathrm{cd}(I, M, R/\mathfrak{p}) = p + d\}.$$

In order to prove Theorem 4 we need the following lemmas.

Lemma 5 *Let M, N be two finitely generated R-modules and t a non-negative integer. If $H^t_{I,J}(M, R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$, then $H^t_{I,J}(M, N) = 0$.*

Proof Since N is a finitely generated R-module, there is a chain of submodules of N

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots \subseteq N_k = N$$

such that $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Supp}_R(N)$. The short exact sequence

$$0 \to N_{i-1} \to N_i \to R/\mathfrak{p}_i \to 0$$

induces $H^t_{I,J}(M, N_i) \cong H^t_{I,J}(M, N_{i-1})$ for $1 \leq i \leq k$ by the assumption. Therefore $H^t_{I,J}(M, N) \cong H^t_{I,J}(M, N_0) = 0$.

We have an immediate consequence of Lemma 5.

Corollary 2 *Let M, N be two finitely generated R-modules and t a non-negative integer. Assume that $H^t_{I,J}(M, R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$ and L is a finitely generated R-module such that $\operatorname{Supp}_R(L) \subseteq \operatorname{Supp}_R(N)$. Then $H^t_{I,J}(M, L) = 0$.*

Lemma 6 *Let M, N be two finitely generated R-modules. Assume that $\operatorname{Ext}^i_R(M, H^j_{I,J}(N)) = 0$ for all $i > p$ or $j > r$. Then*

$$H^{p+r}_{I,J}(M, N) \cong \operatorname{Ext}^p_R(M, H^r_{I,J}(N)).$$

Proof Let $G(-) := \Gamma_{I,J}(-)$ and $F(-) := \operatorname{Hom}_R(M, -)$ be functors from the category of R-modules to itself. Then $FG(-) = \Gamma_{I,J}(M, -)$ and F are left exact. If E is an injective R-module, then

$$R^i F(G(E)) = R^i \operatorname{Hom}_R(M, \Gamma_{I,J}(E)) = 0$$

for all $i > 0$, as $\Gamma_{I,J}(E)$ is an injective R-module. By [16, 10.47] there is a Grothendieck spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_R(M, H^j_{I,J}(N)) \underset{i}{\Longrightarrow} H^{i+j}_{I,J}(M, N).$$

We now consider the homomorphisms of the spectral sequence

$$E_k^{p-k,r+k-1} \to E_k^{p,r} \to E_k^{p+k,r+1-k}.$$

By the assumption, we have $E_2^{i,j} = 0$ for all $i > p$ or $j > r$. Then $E_k^{p-k,r+k-1} = E_k^{p+k,r+1-k} = 0$ for all $k \geq 2$, so

$$E_2^{p,r} = E_3^{p,r} = ... = E_\infty^{p,r}.$$

Thus the proof is completed by showing that $E_\infty^{p,r} \cong H^{p+r}_{I,J}(M, N)$. Indeed, there is a filtration Φ of $H^{p+r} = H^{p+r}_{I,J}(M, N)$ such that

$$0 = \Phi^{p+r+1} H^{p+r} \subseteq \Phi^{p+r} H^{r+d} \subseteq \ldots \subseteq \Phi^1 H^{p+r} \subseteq \Phi^0 H^{p+r} = H^{p+r}_{I,J}(M, N)$$

and

$$E_\infty^{i,p+r-i} \cong \Phi^i H^{p+r}/\Phi^{i+1} H^{p+r},\ 0 \le i \le p+r.$$

By the above argument, we get $E_2^{i,p+r-i} = \operatorname{Ext}_R^i(M, H_{I,J}^{p+r-i}(N)) = 0$ for all $i \neq p$. Hence

$$\Phi^{p+1} H^{p+r} = \Phi^{p+2} H^{p+r} = \ldots = \Phi^{p+r+1} H^{p+r} = 0$$

and

$$\Phi^p H^{p+r} = \Phi^{p-1} H^{p+r} = \ldots = \Phi^0 H^{p+r} = H_{I,J}^{p+r}(M,N).$$

It follows

$$E_\infty^{p,r} \cong \Phi^p H^{p+r}/\Phi^{p+1} H^{p+r} \cong H_{I,J}^{p+r}(M,N).$$

The proof is complete.

By [4, 2.1], if $(R,\mathfrak{m})$ is a local ring and N is finitely generated with $d = \dim N$, then $H_{I,J}^d(N)$ is artinian. On the other hand, $H_{I,J}^i(N) = 0$ for all $i > \dim N$ ([17, 4.7 (1)]). Therefore the following result is a consequence of Lemma 6.

Corollary 3 *([14, 3.1]) Let M, N be two finitely generated modules over a local ring $(R,\mathfrak{m})$. Assume that $p = \mathrm{pd}M < \infty$ and $d = \dim N < \infty$. Then $H_{I,J}^{p+d}(M,N) \cong \operatorname{Ext}_R^p(M, H_{I,J}^d(N))$. Moreover $H_{I,J}^{p+d}(M,N)$ is artinian.*

The supremum of all integers i for which $H_{I,J}^i(M,N) \neq 0$, denoted by $\mathrm{cd}(I,J,M,N)$, is called the cohomological dimension of M, N with respect to (I,J)

$$\mathrm{cd}(I,J,M,N) = \sup\{i \mid H_{I,J}^i(M,N) \neq 0\}.$$

If N is finitely generated, then combining Lemma 6 with [17, 4.7] we conclude that

$$\mathrm{cd}(I,J,M,N) \le \min\{\mathrm{pd}M + \dim N, \mathrm{pd}M + \dim N/JN + 1\}.$$

We also recall that the cohomological dimension of N with respect to a pair of ideals (I,J) is defined as ([4])

$$\mathrm{cd}(I,J,N) = \sup\{i \mid H_{I,J}^i(N) \neq 0\}.$$

Lemma 7 *Let M, N, L be finitely generated R-modules such that $\mathrm{pd}M < \infty$ and $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(N)$. Then $\mathrm{cd}(I,J,M,L) \le \mathrm{cd}(I,J,M,N)$. Moreover $\mathrm{cd}(I,J,M,N) = \mathrm{cd}(I,J,M,L)$ if $\mathrm{Supp}_R(N) = \mathrm{Supp}_R(L)$.*

Proof The proof is similar to that in the proof of [3, Theorem B].

We now can prove Theorem 4.

Proof (Proof of Theorem 4) Note that $H^i_{I,J}(M,N)=0$ for all $i>p+d$. If $H^{p+d}_{I,J}(M,N)=0$, then $\operatorname{cd}(I,J,M,N)<p+d$ and $\operatorname{Att}_R(H^{p+d}_{I,J}(M,N))=\emptyset$. Let $\mathfrak{p}\in\operatorname{Ass}_R(N/JN)$, then $R/\mathfrak{p}$ is J-torsion. It follows from [14, 2.7] that $H^i_{I,J}(M,R/\mathfrak{p})\cong H^i_I(M,R/\mathfrak{p})$ for all $i\geq 0$. Since $\mathfrak{p}\in\operatorname{Supp}_R(N)$, we have $\operatorname{cd}(I,J,M,R/\mathfrak{p})\leq\operatorname{cd}(I,J,M,N)<p+d$ by Corollary 2 and Lemma 7. Hence $H^{p+d}_I(M,R/\mathfrak{p})=0$ and then

$$\{\mathfrak{p}\in\operatorname{Ass}_R(N/JN)\mid\operatorname{cd}(I,M,R/\mathfrak{p})=p+d\}=\emptyset.$$

We can assume $H^{p+d}_{I,J}(M,N)\neq 0$. By Lemma 6, one gets

$$\operatorname{Att}_R(H^{p+d}_{I,J}(M,N))=\operatorname{Att}_R(\operatorname{Ext}^p_R(M,H^d_{I,J}(N))).$$

Since $\operatorname{Ext}^p_R(M,-)$ is a right exact additive functor, there is an isomorphism

$$\operatorname{Ext}^p_R(M,H^d_{I,J}(N))\cong\operatorname{Ext}^p_R(M,R)\otimes_R H^d_{I,J}(N).$$

It follows from Lemma 4(ii) that

$$\operatorname{Att}_R(H^{p+d}_{I,J}(M,N))=\operatorname{Att}_R(H^d_{I,J}(N))\cap\operatorname{Supp}_R\operatorname{Ext}^p_R(M,R).$$

By [5, 2.1], one gets

$$\operatorname{Att}_R(H^d_{I,J}(N))=\operatorname{Att}_R(H^d_I(N/JN)).$$

This implies that

$$\begin{aligned}\operatorname{Att}_R(H^{p+d}_{I,J}(M,N))&=\operatorname{Att}_R(H^d_I(N/JN))\cap\operatorname{Supp}_R\operatorname{Ext}^p_R(M,R)\\&=\operatorname{Att}_R(\operatorname{Ext}^p_R(M,R)\otimes_R H^d_I(N/JN))\\&=\operatorname{Att}_R(\operatorname{Ext}^p_R(M,H^d_I(N/JN))\\&=\operatorname{Att}_R(H^{p+d}_I(M,N/JN)).\end{aligned}$$

We know by [9, 2.3] that

$$\operatorname{Att}_R(H^{p+d}_I(M,N/JN))=\{\mathfrak{p}\in\operatorname{Ass}_R(N/JN)\mid\operatorname{cd}(I,M,R/\mathfrak{p})=p+d\}.$$

Therefore, we can conclude that

$$\operatorname{Att}_R(H^{p+d}_{I,J}(M,N))=\{\mathfrak{p}\in\operatorname{Ass}_R(N/JN)\mid\operatorname{cd}(I,M,R/\mathfrak{p})=p+d\},$$

and the proof is complete.

The following proposition shows a result on the vanishing of top generalized local cohomology modules $H^{p+d}_{I,J}(M,N)$.

Proposition 3 *Let $(R,\mathfrak{m})$ be a local ring and M,N finitely generated R-modules with $\operatorname{pd}M=p<\infty$ and $\dim N=d<\infty$. Assume that $I+J$ is $\mathfrak{m}$-primary and $\dim N=d>0$. Then $H^{p+d}_{I,J}(M,N)=0$ or $H^{p+d}_{I,J}(M,N)$ is not finitely generated.*

Proof It follows from the proof of [14, 3.4] that $H^i_{I,J}(M,N) \cong H^i_{\mathfrak{m},J}(M,N)$ for all $i \geq 0$. Moreover

$$\begin{aligned}\mathrm{Att}_R(H^{p+d}_{\mathfrak{m},J}(M,N)) &\subseteq \mathrm{Att}_R(H^d_{\mathfrak{m},J}(N))\\ &\subseteq \mathrm{Att}_R(H^d_{\mathfrak{m}}(N)) \text{ (by [5, 2.1])}\\ &= \{\mathfrak{p} \in \mathrm{Ass}_R(N) \mid \dim R/\mathfrak{p} = d\}.\end{aligned}$$

Since $\dim N = d > 0$, we have $\mathrm{Att}_R(H^{p+d}_{\mathfrak{m},J}(M,N)) \not\subseteq \{\mathfrak{m}\}$.
If $H^{p+d}_{\mathfrak{m},J}(M,N) \neq 0$ and it is finitely generated, then it has finite length. Hence $\mathrm{Att}_R(H^{p+d}_{\mathfrak{m},J}(M,N)) = \{\mathfrak{m}\}$, which is a contradiction.

Proposition 4 *Let* $(R,\mathfrak{m})$ *be a local ring and* M, N *finitely generated* R*-modules with* $\mathrm{pd}M = p < \infty, \dim N = d < \infty$. *Let* I_1, I_2 *be two proper ideals of* R *such that* $I_1 \subseteq I_2$. *Then* $\mathrm{Att}_R(H^{p+d}_{I_1,J}(M,N)) \subseteq \mathrm{Att}_R(H^{p+d}_{I_2,J}(M,N))$.

Proof From [5, 2.5], there is a surjective homomorphism

$$H^d_{I_2,J}(N) \to H^d_{I_1,J}(N).$$

By applying the right exact functor $\mathrm{Ext}^p_R(M,-)$ to the above homomorphism, we get an exact sequence

$$\mathrm{Ext}^p_R(M, H^d_{I_2,J}(N)) \to \mathrm{Ext}^p_R(M, H^d_{I_1,J}(N)) \to 0.$$

Hence, the assertion follows from 3 and 4(ii).

Lemma 8 *Let* N *be a finitely generated* R*-module and* t *a non-negative integer such that* $\mathrm{cd}(I,J,N) \leq t$. *Then*

$$H^t_{I,J}(N/\mathfrak{a}N) \cong H^t_{I,J}(N)/\mathfrak{a}H^t_{I,J}(N).$$

In particular, if $\mathfrak{a} \in \mathrm{Supp}_R(N)$, *then*

$$H^t_{I,J}(N/\mathfrak{a}N) \cong H^t_{I,J}(R/\mathfrak{a}) \otimes_R N.$$

Proof Let $S = R/\mathrm{Ann}_R(N)$, it follows from [17, 2.7] that there is an isomorphism $H^i_{I,J}(N) \cong H^i_{IS,JS}(N)$ for all $i \geq 0$. Note that $\mathrm{cd}(IS,JS,S) = \mathrm{cd}(I,J,N)$, hence $H^i_{IS,JS}(S) = 0$ for all $i > t$ by the hypothesis. It implies that $H^i_{IS,JS}(N) = 0$ for all S-module N and $i > t$. Thus $H^t_{IS,JS}(-)$ is a right exact functor on the category of S-modules. Now we have

$$\begin{aligned}H^t_{I,J}(N/\mathfrak{a}N) &\cong H^t_{IS,JS}(N/\mathfrak{a}N)\\ &\cong H^t_{IS,JS}(S \otimes_S N/\mathfrak{a}N)\\ &\cong H^t_{IS,JS}(S) \otimes_S N/\mathfrak{a}N\\ &\cong H^t_{IS,JS}(S) \otimes_S (N \otimes_R R/\mathfrak{a})\\ &\cong H^t_{IS,JS}(S \otimes_S N) \otimes_R R/\mathfrak{a}\\ &\cong H^t_{IS,JS}(N) \otimes_R R/\mathfrak{a}\\ &\cong H^t_{I,J}(N) \otimes_R R/\mathfrak{a} \cong H^t_{I,J}(N)/\mathfrak{a}H^t_{I,J}(N).\end{aligned}$$

If $\mathfrak{a} \in \mathrm{Supp}_R(N)$, then $R/\mathfrak{a}$ is also an S-module. From the above proof, we have

$$H^t_{I,J}(N/\mathfrak{a}N) \cong H^t_{IS,JS}(S) \otimes_S (R/\mathfrak{a} \otimes_R N) \cong H^t_{I,J}(R/\mathfrak{a}) \otimes_R N,$$

and the lemma follows.

Combining [17, 4.5] with Lemma 8, we have a result on artinianness of generalized local cohomology modules.

Proposition 5 *Let $(R, \mathfrak{m})$ be a local ring and M, N finitely generated R-modules. Assume that $I + J$ is $\mathfrak{m}$-primary and $p = \mathrm{pd}M, t = \dim N/JN$. Then $H^{p+t}_{I,J}(M, N)/JH^{p+t}_{I,J}(M, N)$ is artinian.*

Proof From 6 we have $H^{p+t}_{I,J}(M, N) \cong \mathrm{Ext}^p_R(M, H^t_{I,J}(N))$, because $H^i_{I,J}(N) = 0$ for all $i > t$. Combining the right exactness of the functor $\mathrm{Ext}^p_R(M, -)$ with 8, we get

$$H^{p+t}_{I,J}(M, N)/JH^{p+t}_{I,J}(M, N) \cong \mathrm{Ext}^p_R(M, H^t_{I,J}(N)/JH^t_{I,J}(N)).$$

By [4, 2.3], $H^t_{I,J}(N)/JH^t_{I,J}(N)$ is artinian and the claim follows.

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